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THEORIES WHOSE RESPLENDENT MODELS ARE HOMOGENEOUS

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ABSTRACT

Buechler proved that if T satisfies certain conditions, then all resplendent models of T are homogeneous. Here the theories whose resplendent models are all homogeneous are characterized as satisfying a pair of conditions (weaker than Buechler's). It follows from the characterization that if all resplendent models of T in some uncountable power are homogeneous, then all resplendent models of T are homogeneous.

§1. Introduction

A structure $\mathfrak{A}$ is said to be *resplendent* if for each sentence $\theta(\boldsymbol{a}, R)$, involving a new relation symbol R and some symbols from $D^c(\mathfrak{A})$, if $\theta(\boldsymbol{a}, R)$ is consistent with $D^c(\mathfrak{A})$, then $\theta(\boldsymbol{a}, R)$ is satisfied in an expansion of $\mathfrak{A}$. A structure $\mathfrak{A}$ is said to be *homogeneous* if for any subset X of $\mathfrak{A}$, if $\bar{\bar{X}} < \bar{\bar{\mathfrak{A}}}$, then any elementary monomorphism from X into $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$. The structure $\mathfrak{A}$ is said to be *$\aleph_0$-homogeneous* if every finite elementary monomorphism extends to an automorphism.

Except in the last section of the paper, all languages are assumed to be finite. Under this assumption, the following three statements are true (none of them holds without the assumption):

LEMMA 1.1. *Every resplendent structure is $\aleph_0$-homogeneous.*

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LEMMA 1.2. *A countable structure is resplendent just in case it is recursively saturated.*

LEMMA 1.3. *A structure $\mathfrak{A}$ (of arbitrary cardinality) is resplendent iff for any r.e. theory T (in a finite language), if T is consistent with $D^c(\mathfrak{A})$, then $\mathfrak{A}$ can be expanded to a model of T.*

Buechler [1] proved that if T satisfies certain conditions then all of the resplendent models of T are homogeneous. A striking consequence of this is the fact that if T has just one resplendent model in some uncountable power, then it has just one resplendent model in each infinite power.

The present paper gives a set of two conditions — one on definability of types and one on ubiquity of indiscernibles — characterizing the theories whose resplendent models are all homogeneous. It follows from this characterization that if all resplendent models of T in some uncountable power are homogeneous, then all of the resplendent models of T are homogeneous.

The proof that the two conditions are sufficient uses ideas from Buechler, of course. However, some of the special consequences of ω-stability that Buechler used are not available here. In particular, where Buechler used the existence of prime models over arbitrary sets and the fact that a type over an arbitrary set is the unique non-forking extension of some stationary type over a finite subset, new machinery is called for. Forking is not mentioned at all.

Some notation and terminology will be mentioned here. If $\boldsymbol{x}$, $\boldsymbol{y}$ are finite sequences, then $\boldsymbol{x}^\frown\boldsymbol{y}$ denotes the *concatenation*; i.e., the results of adding the terms of $\boldsymbol{y}$ to the end of $\boldsymbol{x}$. If $\boldsymbol{a}$ is a finite sequence from a structure $\mathfrak{A}$ and $X \subseteq \mathfrak{A}$, then $\mathrm{tp}(\boldsymbol{a}, X, \mathfrak{A})$ denotes the type realized by $\boldsymbol{a}$ over X in $\mathfrak{A}$. The notation $\mathrm{tp}(\boldsymbol{a}, \mathfrak{A})$ may be used instead of $\mathrm{tp}(\boldsymbol{a}, \emptyset, \mathfrak{A})$. The set of all complete 1-types over X is denoted by S_X. If $\Gamma \in S_X$ and $Y \subseteq X$, then $\Gamma \upharpoonright Y$ denotes the set of formulas $\varphi(\boldsymbol{x}, v)$ in Γ having parameters $\boldsymbol{x}$ in Y. So far, everything is standard, but the next definition is new.

Let $\mathfrak{A}$ be a structure, with $X \subseteq \mathfrak{A}$ and $\Gamma \in S_X$. Let $e \in \omega$, and let $\boldsymbol{c}$ be a finite sequence from $\mathfrak{A}$. Let $\mathfrak{A}'$ be an expansion of $\mathfrak{A}$. Then Γ is said to be *determined* by e and $\boldsymbol{c}$ in $\mathfrak{A}'$ if for all finite $\boldsymbol{x} \in X$, the characteristic function of the restricted type $\Gamma_{\boldsymbol{x}}$ is computed by the eth recursive procedure, using information about $\mathrm{tp}(\boldsymbol{x}^\frown\boldsymbol{c}, \mathfrak{A}')$; i.e., $\chi_{\Gamma \upharpoonright \boldsymbol{x}} = \varphi_e^{\mathrm{tp}(\boldsymbol{x}^\frown \boldsymbol{c}, \mathfrak{A}')}$.

If $X \subseteq \mathfrak{A}$ and $\Gamma \in S_X$, then Γ is said to be *finitely satisfied* in $\mathfrak{A}$ if $\Gamma \upharpoonright Y$ is satisfied on $\mathfrak{A}$ for each finite $Y \subseteq X$. The next lemma gives an alternate definition of homogeneity that is often more useful than the original one (see [3]).

LEMMA 1.4. *A structure $\mathfrak{A}$ is homogeneous iff $\mathfrak{A}$ is $\aleph_0$-homogeneous and for* *any* $X \subseteq \mathfrak{A}$ *and any* $\Gamma \in S_X$, *if* $\bar{\bar{X}} < \bar{\bar{\mathfrak{A}}}$ *and* Γ *is finitely satisfied in* $\mathfrak{A}$, *then* Γ *is satisfied in* $\mathfrak{A}$.

Let T be a stable theory, with $\mathfrak{A}$ a model of T. Let $X \subseteq \mathfrak{A}$, and let I be an infinite set of indiscernibles in $\mathfrak{A}$. The *average type* of I over X, denoted by $\mathrm{Av}(I, X)$, is the set of formulas $\varphi(x, v)$ such that $\mathfrak{A} \models \varphi(x, i)$ for infinitely many $i \in I$. (This type is consistent, by stability.)

Now, here are the two conditions:

(1) *Recursive Definability of Types*

Let $\mathfrak{A}$ be a countable recursively saturated model of T, with $X \subseteq \mathfrak{A}$ and $\Gamma \in S_X$. If Γ is finitely satisfied in $\mathfrak{A}$, then Γ is determined by some $e \in \omega$ and $c \in \mathfrak{A}$.

(2) *Ubiquity of Indiscernibles*

Let $\mathfrak{A}$ be a countable recursively saturated model of T, with $X \subseteq \mathfrak{A}$ and $\Gamma \in S_X$. If Γ is finitely satisfied in $\mathfrak{A}$, then either Γ is realized in $\mathfrak{A}$ or else there is an infinite set of indiscernibles $I \subseteq \mathfrak{A}$ such that $\Gamma = \mathrm{Av}(I, X)$.

The Definability Condition, considered just by itself, turns out to be very strong. If a theory T satisfies Condition 1, then it must be superstable, and if, in addition to satisfying Condition 1, T has only countably many types, then it must be ω-stable. There are theories that satisfy Condition 1 but are not ω-stable. In fact, Nadel and Stavi observed that for the theory of $(Z, +, 1)$, which has uncountably many types, all resplendent models are homogeneous, so both conditions are satisfied.

Section 2 contains some basic lemmas extending the two conditions to uncountable models. Section 3 contains the main result. Section 4 gives a definition of resplendence appropriate for infinite languages. This definition is stronger than the standard one. If this definition is adopted, then Lemmas 1.1, 1.2, and 1.3 hold for structures whose language is infinite. In addition, Buechler's results extend to theories in an infinite language, and so does the characterization of theories whose resplendent models are homogenous.

Section 4 contains a couple of examples. The first is a countable structure $\mathfrak{A}$ that is resplendent under the standard definition, but not under the stronger definition. The structure $\mathfrak{A}$ is not $\aleph_0$-homogeneous, and not recursively saturated. The language of $\mathfrak{A}$ is infinite. The other example is a superstable theory T, in an infinite language, such that T has only countably many types, and T has inhomogeneous models that are resplendent under the stronger definition. The theory T fails to satisfy either of the two conditions, Recursive Definability of Types, or Ubiquity of Indiscernibles.

§2. Lemmas on the two conditions

The first lemma says that the Definability Condition extends to uncountable models. The proof is an obvious Downward Löwenheim–Skolem argument, which will be omitted.

LEMMA 2.1. *Suppose that T is a countable complete theory satisfying the Definability Condition. Let $\mathfrak{A}$ be an uncountable recursively saturated model of T, with $X \subseteq \mathfrak{A}$ and $\Gamma \in S_X$. Suppose that Γ is finitely satisfied in $\mathfrak{A}$. Then Γ is determined by some $e \in \omega$ and $c \in \mathfrak{A}$.*

Next it will be shown that the Definability Condition implies superstability.

LEMMA 2.2. *Suppose that T satisfies Condition 1. Let $\mathfrak{A}$ be an ω-saturated model of T. Then if $\bar{\bar{\mathfrak{A}}} = \kappa$, there are only κ types over $\mathfrak{A}$. Hence, T is necessarily superstable, and if T has only countable many types, then T is ω-stable.*

PROOF. Each type over $\mathfrak{A}$ is determined by some finite $c \in \mathfrak{A}$ and some $e \in \omega$.

It is not clear whether Condition 1 implies Condition 2. Certainly, if $\Gamma \in S_X$ and Γ is finitely satisfied in $\mathfrak{A}$ but not actually realized in $\mathfrak{A}$, there is a set of indiscernibles I, is some "monster" model of T, such that $\Gamma = \mathrm{Av}(I, X)$. The content of Condition 2 is that I can be taken to be a subset of $\mathfrak{A}$.

Lemma 2.1 says that if Condition 1 holds for countable models, then it holds for uncountable models. Condition 2 also extends to uncountable models, but proving this requires more of an argument, and Condition 1 is assumed.

LEMMA 2.3. *Suppose that T is a countable complete theory satisfying Conditions 1 and 2 (for countable models). Let $\mathfrak{A}$ be an uncountable recursively saturated model of T, and let Γ be a type over a set $X \subseteq \mathfrak{A}$, where Γ is finitely satisfied in $\mathfrak{A}$. If Γ is not realized in $\mathfrak{A}$, then there is a set of indiscernibles $I \subseteq \mathfrak{A}$ such that $\Gamma = \mathrm{Av}(I, X)$.*

PROOF. Assume there is no such set of indiscernibles. Take $\mathfrak{A}_0$ to be a countable recursively saturated elementary substructure of $\mathfrak{A}$ such that if $X_0 = X \cap \mathfrak{A}_0$, then $\Gamma \restriction X_0$ is finitely satisfied on $\mathfrak{A}_0$. Then by Condition 2, there is a set of indiscernibles $I_0 \subseteq \mathfrak{A}_0$ such that $\mathrm{Av}(I_0, X_0) = \Gamma \restriction X_0$. By assumption, $\mathrm{Av}(I_0, X) \neq \Gamma$. Take some finite sequence $y_0 \in X$ witnessing this fact, and let $\mathfrak{A}_1$ be another countable recursively saturated model such that $\mathfrak{A}_0 < \mathfrak{A}_1 < \mathfrak{A}$, $y_0 \in \mathfrak{A}_1$, and if $X_1 = X \cap \mathfrak{A}_1$, then $\Gamma \restriction X_1$ is finitely satisfied in $\mathfrak{A}_1$. As before, there is a set of indiscernibles $I_1 \subseteq \mathfrak{A}_1$ such that $\mathrm{Av}(I_1, X_1) = \Gamma \restriction X_1$, and there is some $y_1 \in X$ such that $\mathrm{Av}(I_1, y_1) \neq \Gamma \restriction y_1$.

Continue choosing $\mathfrak{A}_\alpha$, X_α, I_α, and y_α for $\alpha < \omega_1$. At limit stages, let $\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$. (Note that for $X_\alpha = X \cap \mathfrak{A}_\alpha$, the type $\Gamma \restriction X_\alpha$ is finitely satisfied in $\mathfrak{A}_\alpha$, provided that for each $\beta < \alpha$, $\Gamma \restriction X_\beta$ was finitely satisfied in $\mathfrak{A}_\beta$.) By the Definability Condition, for each $\alpha < \omega_1$, there exist $e_\alpha \in \omega$ and $c_\alpha \in \mathfrak{A}_\alpha$ determining $\mathrm{Av}(I_\alpha, \mathfrak{A}_\alpha)$. By Fodor's Theorem, there exist a fixed e and c that work for a stationary set of α's. Now let $\mathfrak{A}_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$. For each $a \in \mathfrak{A}_{\omega_1}$ and for all α from the stationary set, if $a \in \mathfrak{A}_\alpha$, then $\varphi_e^{\mathrm{tp}(a^\frown c, \mathfrak{A})} = \mathrm{Av}(I_\alpha, a)$.

With a little further effort, it is possible to show that for α, β in the stationary set, if $\alpha < \beta$, then $\mathrm{Av}(I_\alpha, \mathfrak{A}_\beta) = \mathrm{Av}(I_\beta, \mathfrak{A}_\beta)$. Let $a \in \mathfrak{A}_\beta$. For any formula φ, it must be shown that $\varphi(a, v) \in \mathrm{Av}(I_\alpha, \mathfrak{A}_\beta)$ iff $\varphi(a, v) \in \mathrm{Av}(I_\beta, \mathfrak{A}_\beta)$. By stability, there is some $n \in \omega$ such that for any set of indiscernibles I in the monster model $\mathcal{M}$, there cannot be $I_1, I_2 \subseteq I$, both of size n, such that for all $i \in I_1$, $\mathcal{M} \models \varphi(a, i)$ and for all $i \in I_2$, $\mathcal{M} \models \sim \varphi(a, i)$. Suppose that $\varphi(a, v) \in \mathrm{Av}(I_\beta, a)$. Let $i = (i_0, \cdots, i_{n-1})$ be a sequence of distinct elements of I_α such that $\mathrm{tp}(i_k, c^\frown i \restriction k, \mathfrak{A})$ is determined by e and c. Then take a sequence $j = (j_k)_{k\in\omega}$ of distinct elements of I_β such that $\mathrm{tp}(j_k, c^\frown a^\frown i^\frown j \restriction k, \mathfrak{A})$ is determined by e and c. Combine these to form the set of indiscernibles $J = \mathrm{ran}\, i \cup \mathrm{ran}\, j$. Then $\varphi(a, v) \in \mathrm{Av}(J, a) = \mathrm{Av}(I_\beta, a)$. This proves that $\mathrm{Av}(I_\beta, \mathfrak{A}_\beta) = \mathrm{Av}(I_\alpha, \mathfrak{A}_\beta)$, which is a contradiction, since it was assumed that $\mathrm{Av}(I_\alpha, y_\alpha) \neq \Gamma \restriction y_\alpha$, where $y_\alpha \in \mathfrak{A}_{\alpha+1} < \mathfrak{A}_\beta$, but $\mathrm{Av}(I_\beta, y_\alpha) = \Gamma \restriction y_\alpha$. Therefore, there must be some infinite set of indiscernibles $I \subseteq \mathfrak{A}$ such that $\mathrm{Av}(I, X) = \Gamma$.

§3. An Omitting Types Theorem, and the main result

Next, it will be shown that the two conditions are sufficient for all resplendent models of T to be homogeneous.

LEMMA 3.1. *Suppose that T is a countable complete theory satisfying Conditions 1 and 2. If $\mathfrak{A}$ is an uncountable resplendent model of T, then $\mathfrak{A}$ is homogeneous. (Of course, a countable resplendent model is always homogeneous, under the assumption that the language is finite.)*

PROOF. Let $X \subseteq \mathfrak{A}$, where $\bar{\bar{X}} < \bar{\bar{\mathfrak{A}}}$. Let $\Gamma \in S_X$, where Γ is finitely satisfied in $\mathfrak{A}$. It will be shown that Γ is realized in $\mathfrak{A}$. (By Lemmas 1.1 and 1.4, this is sufficient to prove homogeneity.) If Γ is not realized in $\mathfrak{A}$, then $\Gamma = \mathrm{Av}(I, X)$ for some infinite set I of indiscernibles in $\mathfrak{A}$. Moreover, $\mathrm{Av}(I, X)$ is determined by some $x \in \mathfrak{A}$ and some $e \in \omega$.

The model $\mathfrak{A}$ will be expanded to a model of a certain r.e. theory, in a language with a new unary relation symbol J, a new binary relation symbol F, and constants for the finite sequence of elements c, in addition to the symbols from the language of $\mathfrak{A}$. This r.e. theory says that J is a set of indiscernibles (for the language of $\mathfrak{A}$), F maps J one-one onto the universe, and for all finite sequences $\boldsymbol{i} \in J$ and all $j \in J - \boldsymbol{i}$, $\mathrm{tp}(j, \boldsymbol{i}, \mathfrak{A})$ is determined by e and $\boldsymbol{c}$; i.e., for $\alpha(\boldsymbol{c}, \boldsymbol{u})$ a finite set of formulas, if

$\varphi_e^{\alpha(c,u)}(\gamma(\boldsymbol{u}, v)) = 1$, then $\alpha(\boldsymbol{c}, \boldsymbol{i}) \rightarrow \gamma(\boldsymbol{i}, j)$ holds, and if

$\varphi_e^{\alpha(c,u)}(\gamma(\boldsymbol{u}, v)) = 0$, then $\alpha(\boldsymbol{c}, \boldsymbol{i}) \rightarrow \sim \gamma(\boldsymbol{i}, j)$ holds.

By Lemma 1.3, the fact that $\mathfrak{A}$ is resplendent means that it can be expanded to a model of this theory. Let the set of indiscernibles be called J (to match the symbol). Then $\mathrm{Av}(J, X) = \Gamma$. (If there is any doubt about this, see the part in the proof of Lemma 2.3 where it was shown that $\mathrm{Av}(I_\alpha, \mathfrak{A}_\beta) = \mathrm{Av}(I_\beta, \mathfrak{A}_\beta)$, given that the same $\boldsymbol{c}$ and e determine $\mathrm{Av}(I_\alpha, \mathfrak{A}_\beta)$ and $\mathrm{Av}(I_\beta, \mathfrak{A}_\beta)$.)

The next lemma is an Omitting Types Theorem, where the types may have infinitely many parameters and the models are recursively saturated.

Lemma 3.2. *Let $\mathfrak{A}$ be a countable recursively saturated structure. Let $X \subseteq \mathfrak{A}$, and let $\Gamma \in S_X$. Let $\mathfrak{A}'$ be a recursively saturated expansion of $\mathfrak{A}$, and suppose that Γ is not determined by any e and $\boldsymbol{c}$ in $\mathfrak{A}'$. Then for any r.e. theory T that is consistent with $D^c(\mathfrak{A}')$, there is a recursively saturated expansion $\mathfrak{A}''$ of $\mathfrak{A}'$ to a model of T such that Γ is not determined by any e and $\boldsymbol{c}$ in $\mathfrak{A}''$.*

Proof. The usual method of expanding recursively saturated models is in stages, where at each stage, an r.e. set of sentences $T(\boldsymbol{a}) \supseteq T$ has been determined. Only finitely many constants $\boldsymbol{a}$ from $\mathfrak{A}'$ appear in $T(\boldsymbol{a})$, and the consequences of $T(\boldsymbol{a})$ in the language of $\mathfrak{A}'$ with constants $\boldsymbol{a}$ added are satisfied in $(\mathfrak{A}', a)$.

Suppose that the construction has reached the stage where something must be done to make sure that Γ will not be determined in $\mathfrak{A}''$ by some particular $e \in \omega$ and sequence $\boldsymbol{c}$. Consider finite sets of $L(\mathfrak{A}'')$-formulas α. The plan is to add to $T(\boldsymbol{a})$ one of the following, for some $\boldsymbol{x} \in X$ and some $L(\mathfrak{A})$-formula $\gamma(\boldsymbol{u}, v)$:

(1) $\bigwedge\!\!\!\bigwedge \alpha(\boldsymbol{c}, \boldsymbol{x})$, where $\varphi_e^{\alpha(c,u)}(\gamma(\boldsymbol{u}, v)) = 1$, and $\gamma(\boldsymbol{x}, v) \notin \Gamma$,

(2) $\bigwedge\!\!\!\bigwedge \alpha(\boldsymbol{c}, \boldsymbol{x})$, where $\varphi_e^{\alpha(c,u)}(\gamma(\boldsymbol{u}, v)) = 0$, and $\gamma(\boldsymbol{x}, v) \in \Gamma$,

(3) the set of all sentences $\sim \bigwedge\!\!\!\bigwedge \alpha(\boldsymbol{c}, \boldsymbol{x})$ such that $\varphi_e^{\alpha(c,u)}(\gamma(u, v))$ converges.

Then $\gamma(\boldsymbol{x}, v) \notin \Gamma$ and $\varphi_e^{\mathrm{tp}(x \frown c, \mathfrak{A}'')}(\gamma(\boldsymbol{x}, v)) = 1$, or $\gamma(\boldsymbol{x}, v) \in \Gamma$ and $\varphi_e^{\mathrm{tp}(x \frown c, \mathfrak{A}'')}(\gamma(\boldsymbol{x}, v)) = 0$, or $\varphi_e^{\mathrm{tp}(x \frown c, \mathfrak{A}'')}(\gamma(\boldsymbol{x}, v))$ is undefined. The proof that it is possible to add something of one of the three forms above is by contradiction. It will be shown that if nothing like this can be added, then Γ is already determined in $\mathfrak{A}'$ by $\boldsymbol{a} \frown \boldsymbol{c}$ and some $e' \in \omega$. The recursive procedure with index e' will be described in terms of proofs in an infinite language (the use of the infinite language is not essential).

For each $L(\mathfrak{A})$-formula $\gamma(u, v)$, let R_γ be a new relation symbol, with places for the variables $\boldsymbol{u}$. Let $\Lambda(c, x)$ be the set of all sentences of the forms $\bigwedge\!\!\!\bigwedge \alpha(\boldsymbol{c}, \boldsymbol{x}) \rightarrow R_\gamma(\boldsymbol{x})$, where $\varphi_e^{\alpha(c,u)}(\gamma(\boldsymbol{u}, v)) = 1$, and $\bigwedge\!\!\!\bigwedge \alpha(\boldsymbol{c}, \boldsymbol{x}) \rightarrow \sim R_\gamma(\boldsymbol{x})$, where $\varphi_e^{\alpha(c,u)}(\gamma(\boldsymbol{u}, v)) = 0$. Then Γ is determined in the following way: for each $\boldsymbol{x} \in X$, each $L(\mathfrak{A})$-formula $\gamma(u, v)$, look at the list of theorems proved from $\mathrm{tp}(\boldsymbol{a}^\frown \boldsymbol{c}^\frown \boldsymbol{x}, \mathfrak{A}') \cup T(\boldsymbol{a}) \cup \Lambda(\boldsymbol{c}, \boldsymbol{x})$. If $R_\gamma(\boldsymbol{x})$ appears first, then $\gamma(\boldsymbol{x}, v) \in \Gamma$, and if $\sim R_\gamma(\boldsymbol{x})$ appears first, then $\gamma(\boldsymbol{x}, v) \notin \Gamma$. This shows that it is possible to carry out the steps in the construction so that Γ will not be determined in $\mathfrak{A}''$.

The next lemma is really the second half of the main result.

LEMMA 3.3. *Let $\lambda > \aleph_0$, and suppose that T is a theory whose resplendent models of power λ are all homogeneous. Then T satisfies Conditions* 1 *and* 2.

PROOF. Let $\mathfrak{A}$ be a countable recursively saturated model, with $X \subseteq \mathfrak{A}$, $\Gamma \in S_X$, where Γ is finitely satisfied in $\mathfrak{A}$. If $\mathfrak{A} < \mathfrak{B}$ where $\mathfrak{B}$ is a resplendent model of power λ, then Γ will be realized in $\mathfrak{B}$ (by homogeneity). Using a technique of Schmerl, it can be shown that if Condition 1 fails, then there is a resplendent model $\mathfrak{B}$ of power λ such that $\mathfrak{A} < \mathfrak{B}$ and $\mathfrak{B}$ omits Γ.

Schmerl proved that if $\mathfrak{A}$ is a countable recursively saturated structure, then for any infinite cardinal λ, there is a resplendent model $\mathfrak{B}$ of power λ such that $\mathfrak{B} \equiv_{\infty\omega} \mathfrak{A}$. (See [1] or [2] for the proof.) A slight modification of Schmerl's construction is needed to produce the model $\mathfrak{B}$ that omits Γ, so the construction will be outlined here.

The first step is to expand $\mathfrak{A}$ to a recursively saturated model $\mathfrak{A}_0$ of first order Peano arithmetic. Then the structure is expanded to further recursively saturated models witnessing the resplendence of $\mathfrak{A}$. Let $\mathfrak{A}_n$ be the result of the first n expansions, and let $\mathfrak{A}^*$ be the result of adding everything. Each $\mathfrak{A}_n$ is recursively saturated, but $\mathfrak{A}^*$ is not. By Lemma 3.2, the structures $\mathfrak{A}_n$ can be chosen such that Γ is not determined by any e and $\boldsymbol{c}$ in $\mathfrak{A}_n$.

The next step is to form a nested sequence of infinite sets $(P_i)_{i\in\omega}$, such that each P_i is a unary predicate from $\mathfrak{A}^*$. These sets serve as approximations to a special kind of set of indiscernibles that will generate the resplendent models of arbitrary cardinality. The P_i's have the following properties:

A. For each $L(\mathfrak{A}^*)$-term $\tau(\boldsymbol{u})$, there is some k such that for all $L(\mathfrak{A})$-formulas φ, all increasing sequences $\boldsymbol{a}, \boldsymbol{b} \in P_k$, $\mathfrak{A}^* \models \varphi(\tau(\boldsymbol{a}) \leftrightarrow \varphi(\tau(\boldsymbol{b}))$.

B. For each $L(\mathfrak{A}^*) \cup \{R\}$-formula $\theta(\boldsymbol{u}, R)$, there is some k such that either (a) there is an $L(\mathfrak{A}^*)$-formula $\sigma(\boldsymbol{u})$ such that $\theta(\boldsymbol{u}, R) \vdash \sigma(\boldsymbol{u})$ and for all increasing sequences $\boldsymbol{a} \in P_k$, $\mathfrak{A}^* \models \sim \sigma(\boldsymbol{a})$, or else (b) there is some R in $\mathfrak{A}^*$ such that for all increasing sequences $\boldsymbol{a} \in P_k$, if $R_{\boldsymbol{a}} = \{y : (\boldsymbol{a}, y) \in R\}$, then $\mathfrak{A}^* \models \theta(\boldsymbol{a}, R_{\boldsymbol{a}})$.

C. For each $L(\mathfrak{A}^*)$-term $\tau(\boldsymbol{u})$, there is some k such that for some $\gamma(\boldsymbol{x}, v) \in \Gamma$, for all increasing sequences $\boldsymbol{a} \in P_k$, $\mathfrak{A}^* \models \sim \gamma(\boldsymbol{x}, \tau(\boldsymbol{a}))$.

Properties A and B are the same as in Schmerl's construction. Property C has been added to keep the large resplendent models from realizing the type Γ. Suppose that P_k has been determined, and it is time to find P_{k+1} satisfying Property C for some $\tau(\boldsymbol{u})$. If the desired P_{k+1} did not exist, then it would be possible to test whether a formula $\gamma(\boldsymbol{x}, v)$ is in Γ by the following prodecure:

Take the first n such that there do not exist sets S_1, S_2 of size n such that $S_1 \cup S_2 \subseteq P_k$, for all increasing sequences $\boldsymbol{a} \in S_1$, $\mathfrak{A}^* \models \gamma(\boldsymbol{x}, \tau(\boldsymbol{a}))$, and for all increasing sequences $\boldsymbol{a} \in S_2$, $\mathfrak{A}^* \models \sim \gamma(\boldsymbol{x}, \tau(\boldsymbol{a}))$. If there is no such S_1, then $\gamma(\boldsymbol{x}, v) \notin \Gamma$, and if there is no such S_2, then $\gamma(\boldsymbol{x}, v) \in \Gamma$. This means that Γ is determined by the constants $\boldsymbol{c}$ and an appropriate $e \in \omega$, in any $\mathfrak{A}_n$ containing the relations used in building up τ and P_k. This is a contradiction. Therefore, the desired P_{k+1} exists.

Let $\mathfrak{C}^*$ be a model of $D^c(\mathfrak{A}^*)$ with a set I of size λ such that for any $L(\mathfrak{A}^*)$-formula $\varphi(\boldsymbol{u})$, if there is some P_k such that $\mathfrak{A}^* \models \varphi(\boldsymbol{a})$ for all increasing sequences $\boldsymbol{a} \in P_k$, then $\mathfrak{A}^* \models \varphi(\boldsymbol{i})$ for all increasing sequences $\boldsymbol{i} \in I$. Let $\mathfrak{L}^*$ be the Skolem hull of I in $\mathfrak{C}^*$, and let $\mathfrak{L}$ be the $L(\mathfrak{A})$-reduct of $\mathfrak{L}^*$. Then $\mathfrak{L}$ is resplendent, and it may be assumed that $\mathfrak{A} < \mathfrak{L}$. Property C guarantees that $\mathfrak{L}$ omits Γ. This completes the proof that T satisfies the Definability Condition.

Now, it will be shown that if all resplendent models of T are homogeneous, then T satisfies the condition on Ubiquity of Indiscernibles. Let $\mathfrak{A}$ be a countable recursively saturated model of T, with $X \subseteq \mathfrak{A}$ and $\Gamma \in S_X$, and suppose that Γ is finitely satisfied in $\mathfrak{A}$. It must be shown that if Γ is not realized in $\mathfrak{A}$, then Γ is the average type over X of some set of indiscernibles in $\mathfrak{A}$. Schmerl's construction is used to obtain the set of indiscernibles.

In the construction, the sequence of sets $(P_k)_{k \in \omega}$ is chosen to satisfy Properties A and B, as before, but Property C is replaced by the following:

C′. For each $n \in \omega$, there is some k such that P_k is a set of $L(\mathfrak{A}_n)$-indiscernibles.

Again the set I is taken to have power λ and to satisfy the types determined by the sequence $(P_k)_{k \in \omega}$. If $\mathfrak{L}^*$ is the Skolem hull of I, and $\mathfrak{L}$ is the $L(\mathfrak{A})$-reduct of $\mathfrak{L}^*$, then $\mathfrak{L}$ is resplendent, and it may be assumed that $\mathfrak{A} < \mathfrak{L}$. Since $\mathfrak{L}$ is homogeneous, and the type Γ is finitely satisfied in $\mathfrak{L}$, $\mathfrak{L}$ must realize Γ. Say $\tau^{\mathfrak{L}^*}(\boldsymbol{i})$ realizes Γ, where τ is an $L(\mathfrak{A}_n)$-term, and $\boldsymbol{i}$ is a sequence from I of length r. Condition C′ says that some P_k is a set of $L(\mathfrak{A}_n)$-indiscernibles. Form an infinite sequence of finite sequences $(\boldsymbol{a}_j)_{j \in \omega}$, where each $\boldsymbol{a}_j$ is an increasing sequence of length r in $P_k \cap P_j$ and $\sup(\boldsymbol{a}_j) < \inf(\boldsymbol{a}_{j+1})$, for each $j \in \omega$. For $r = 3$, the arrangement looks like this:

$$\begin{array}{ccccccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{a}_0 & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \cdot & \cdot & \cdot \end{array}$$

If Γ is not realized in $\mathfrak{A}$, then τ is not constant on the $\boldsymbol{a}_j$'s. In fact, the set $\{\tau^{\mathfrak{A}^*}(\boldsymbol{a}_j) : j \in \omega\}$ is a set of $L(\mathfrak{A})$-indiscernibles whose average type over X is Γ, since for any $\boldsymbol{x} \in X$, there is some j such that for all increasing sequences $\boldsymbol{a} \in P_j$, $\mathrm{tp}(\tau^{\mathfrak{A}^*}(\boldsymbol{a}), \boldsymbol{x}, \mathfrak{A}) = \mathrm{tp}(\tau^{\mathfrak{L}^*}(\boldsymbol{i}), \boldsymbol{x}, \mathfrak{L}) = \Gamma \restriction x$. Therefore, the condition on Ubiquity of Indiscernibles is satisfied.

Here is the main result.

THEOREM 3.4. *Let T be a countable complete theory. Then the following are equivalent*:

(a) *T satisfies Conditions* (1) *and* (2),

(b) *all resplendent models of T are homogeneous,*

(c) *all resplendent models of T in some uncountable power λ are homogeneous.*

PROOF. By Lemma 3.1, (a) implies (b). Obviously, (b) implies (c). Finally, by Lemma 3.3, (c) implies (a).

§4. Infinite languages

Let $\mathscr{L}$ be a fixed universal language, with infinitely many constants, and infinitely many m-placed relation and function symbols for each m. Assume, moreover, that the relations "s is the kth constant," "s is the kth m-placed relation symbol," and "s is the kth m-placed function symbol" are all recursive. Consider structures $\mathfrak{A}$ whose language is a recursive, but possibly infinite, subset of $\mathscr{L}$. The following definition of resplendence seems appropriate for these structures: $\mathfrak{A}$ is *resplendent* if for any r.e. theory T, in a language with finitely many new symbols in addition to those of the language of $\mathfrak{A}$, if T is consistent with $D^c(\mathfrak{A})$, then $\mathfrak{A}$ can be expanded to a model of T. This "fixed-language" approach was suggested to the author by Baldwin, Blass, and Lachlan.

If this definition is adopted, then resplendent structures will be recursively saturated and $\aleph_0$-homogeneous. Schmerl's theorem holds; that is, if $\mathfrak{A}$ is a countable recursively saturated structure, whose language is a recursive subset of $\mathscr{L}$, then for each $\kappa > \aleph_0$, there is some $\mathfrak{L}$ of power κ such that $\mathfrak{L} \equiv_{\infty\omega} \mathfrak{A}$ and $\mathfrak{L}$ satisfies the stronger definition of resplendence. The theories T (whose language is a recursive subset of $\mathscr{L}$) with no inhomogeneous models that satisfy this definition of resplendence are characterized by Conditions 1 and 2 of this paper.

The following structure is resplendent by the standard definition, but is not $\aleph_0$-homogeneous or recursively saturated. Let $\mathfrak{A}$ be a countable structure with an equivalence relation that splits the universe into two infinite parts, and with a nested family of unary relations $U_n^{\mathfrak{A}}$, for $n \in \omega$, such that $U_0^{\mathfrak{A}} = \mathfrak{A}$, and $U_n^{\mathfrak{A}} - U_{n+1}^{\mathfrak{A}}$ contains infinitely many elements from each of the two equivalence classes. Let the intersection of the $U_n^{\mathfrak{A}}$'s be non-empty and lie entirely in one equivalence class. The structure $\mathfrak{A}$ satisfies the standard definition of resplendence, since each reduct to a finite language is saturated. However, there is no automorphism that moves an element from one equivalence class to the other, even though the elements of $U_n^{\mathfrak{A}} - U_{n+1}^{\mathfrak{A}}$ all satisfy the same 1-type. This structure can certainly be thought of as satisfying the language requirement above. It does not satisfy the stronger definition of resplendence.

The next example is a theory that is superstable but does not satisfy either Condition 1 or Condition 2. The example was developed in the course of a conversation with Buechler, Kaufmann, and Kueker. Let $\mathfrak{A}_0 = (2^{<\omega}, \sim_n)_{n \in \omega}$, where for $\sigma, \tau \in 2^{<\omega}$, $\sigma \sim_n \tau$ iff $\sigma \restriction n = \tau \restriction n$. Let $T = \mathrm{Th}(\mathfrak{A}_0)$. It is not difficult to verify that T is superstable and that all countable recursively saturated models of T are isomorphic. The Definability Condition must fail, since T is not ω-stable but has only countably many types.

If $\mathfrak{A}$ is a model of T, there is a natural equivalence relation on $\mathfrak{A}$ given by the following condition: For $a, b \in \mathfrak{A}$, $a \sim_\omega b$ iff $a \sim_n^{\mathfrak{A}} b$, for all $n \in \omega$. The condition on Ubiquity of Indiscernibles must fail because any infinite set of indiscernibles must lie entirely in one $\sim_\omega$-class. If $\mathfrak{A}$ is a countable recursively saturated model, then there is some type $\Gamma \in S_{\mathfrak{A}}$ such that Γ is finitely satisfied in $\mathfrak{A}$ but Γ is not the average type of a set of indiscernibles in any $\sim_\omega$-class represented in $\mathfrak{A}$.

The language of T is infinite, but it can be thought of as satisfying the requirement above. There are uncountable models of T which are inhomogeneous but satisfy the stronger definition of resplendence. Specifically, Schmerl's construction, with a set of indiscernibles of order type ω_1, yields a model $\mathfrak{A}$ of power $\aleph_1$ that is resplendent in the strong sense and has only $\aleph_0$ $\sim_\omega$-classes. The model $\mathfrak{A}$ has a countable recursively saturated elementary submodel $\mathfrak{B}$ with representatives from all $\sim_\omega$-classes in $\mathfrak{A}$. There is an elementary embedding f of $\mathfrak{B}$ properly into itself, and f cannot be extended to an automorphism of $\mathfrak{A}$.

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